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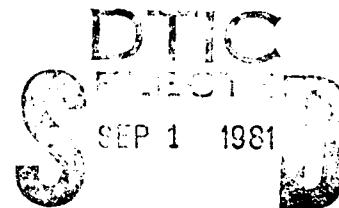
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# AN OPTIMAL PROPERTY OF $S^2$ CHARTS

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## O. ABSTRACT

Let  $\tau_1$  be the class of all procedures for monitoring the variance of a process at level  $\alpha$  using control charts based on statistics  $\{T(Y_1), T(Y_2), \dots\}$  from independent random samples. Suppose the control variance  $\sigma_0^2$  is known. Under Gaussian assumptions the  $S^2$  chart using the sample variance is shown to be optimal in the class  $\tau_1$  in that its run length is stochastically smallest under both stationary and drifting processes not in control. Weaker properties are given in terms of stochastic bounds when  $\sigma_0^2$  is not known and instead is estimated in a base period using the sample variance.

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## 1. INTRODUCTION

Various control charts are used in monitoring the parameters of a production process. For monitoring the variance of product quality the standard procedures are the R and  $S^2$  charts using the sample range and the sample variance, respectively. In these charts successive values of a statistic are plotted against time, and a chart signals at level  $\alpha$  that the monitored process is not in control when the statistic exceeds a control limit  $c_\alpha$ . The standard assumptions are that (i) the process is stationary in its variance with value  $\sigma^2$ , (ii) the control value  $\sigma_0^2$  is known, (iii) independent random samples of n observations each are taken on successive monitoring occasions, and (iv) the sampled process is Gaussian. For the R and  $S^2$  charts the statistics are  $\{R_i/\sigma_0; i = 1, 2, \dots\}$  and  $\{S_i^2/\sigma_0^2; i = 1, 2, \dots\}$ , respectively, where  $R_i$  and  $S_i^2$  are the sample range and the sample variance of the ith sample. An account of these procedures is given in Duncan (1974).

On each sampling occasion the problem is to test at level  $\alpha$  the hypothesis  $H: \sigma^2 = \sigma_0^2$  against  $A: \sigma^2 > \sigma_0^2$ , for which  $S^2$  is known to have greater power than R. The usual notions of level and power do not carry over to the entire monitoring period, however, as the standard charts eventually signal with unit probability even when the process is in control. What is required is that a chart should signal infrequently when the process is in control and more frequently otherwise. Such properties are summarized in the distribution of the run length of a chart, i.e. the number of successive samples taken before the chart signals. Run lengths accordingly provide a mechanism for comparing two charts,

for if the run length of one chart is stochastically smaller than another when the process is not in control, then that chart will tend to signal more quickly. Under standard assumptions the run lengths of the R and  $S^2$  charts have geometric distributions, each with parameter equal to the probability of exceeding the control limit on any sampling occasion; otherwise the distributions are more complicated.

Although tradition mandates use of the R or  $S^2$  chart, clearly any of numerous statistics could be used in monitoring the variance of a Gaussian process. Here we show that the known optimality of  $S^2$  in testing H against A on any sampling occasion carries over to the  $S^2$  chart in the following sense. Among all  $\alpha$ -level monitoring procedures, the run length of the  $S^2$  chart is stochastically minimal when the process is not in control. Weaker results provide stochastically smallest bounds when  $\sigma_0^2$  is not known and instead is estimated using the sample variance from a base period.

## 2. THE MAIN RESULTS

Let  $\{Y_1, Y_2, \dots\}$  be n-dimensional observations taken on successive occasions; let  $\{T(Y_1), T(Y_2), \dots\}$  be the values of a statistic used in monitoring the variance; and let  $N_T$  be the run length of the corresponding chart. Suppose the chart signals at level  $\alpha$  whenever  $T(Y_i) \in B$ , i.e.  $P(T(Y_i) \in B) = \alpha$  when the process is in control. It is clear that

$$P(N_T > k) = P(T(Y_1) \in B^c, \dots, T(Y_k) \in B^c) \quad (2.1)$$

with  $B^c$  as the complement of B. Under the assumption that  $\{T(Y_1), T(Y_2), \dots\}$  are mutually independent, (2.1) becomes

$$P(N_T > k) = \prod_{i=1}^k (1 - \beta_i) \quad (2.2)$$

with  $\beta_i = P(T(Y_i) \in B)$ . For nonstationary processes probabilities of the type (2.2) depend on a countable sequence  $\beta = (\beta_1, \beta_2, \dots)$  which we call the drifting parameters of the process. Accordingly, let  $F_T(u; \beta) = 1 - P(N_T > u)$  be the cdf of  $N_T$  under drifting. For stationary processes  $\{T(Y_1), T(Y_2), \dots\}$  are distributed identically, in which case the cdf of  $N_T$  is the geometric distribution  $G(k; \beta) = 1 - (1 - \beta)^k$  with  $\beta = P(N_T \in B)$ , i.e.  $F(u; \beta) = G(u; \beta)$  with  $\beta = (\beta, \beta, \dots)$ . Let  $\mathcal{D}$  be the collection of all sequences  $\delta = (\delta_1, \delta_2, \dots)$  such that  $\delta_i \in (0, 1)$  for all  $i$ , and let  $\{F_T(u; \beta); \beta \in \mathcal{D}\}$  be the family of possible run-length distributions associated with a given chart. Any such family is said to decrease stochastically in  $\beta$  if, for any  $\beta$  and  $\beta^*$  in  $\mathcal{D}$  such that  $\beta_i^* \geq \beta_i$  for all  $i$ ,  $F(u; \beta^*) \geq F(u; \beta)$  for every  $u > 0$ . From the form of (2.2) it is clear that  $\{F_T(u; \beta); \beta \in \mathcal{D}\}$  decreases stochastically in  $\beta$  for any chart of the type considered here.

Let  $\tau_1$  be the class of all  $\alpha$ -level procedures for monitoring the variance of a process using statistics  $\{T(Y_1), T(Y_2), \dots\}$ . Our result for this class is that under Gaussian assumptions the  $S^2$  chart is optimal in that its run length is stochastically minimal when the process is not in control.

THEOREM 1. In the class  $\tau_1$  of all procedures for monitoring the variance of a drifting Gaussian process at level  $\alpha$  using statistics of the type  $\{T(Y_1), T(Y_2), \dots\}$ , the  $S^2$  chart is optimal in that its run length is stochastically minimal.

Proof. Consider any procedure  $T$  in  $\tau_1$  together with its run-length distribution  $F_T(u; \beta)$ , and let  $F_S(u; \beta^*)$  be the run-length distribution of the  $S^2$  chart. Considered as procedures for testing  $H: \sigma^2 = \sigma_0^2$  against  $A: \sigma^2 > \sigma_0^2$  on each sampling occasion, under Gaussian assumptions the test using  $S_i^2/\sigma_0^2$  is uniformly most powerful among procedures in  $\tau_1$  (cf. Lehmann (1959), page 97), thus assuring that  $\beta_i^* \geq \beta_i$  for all  $i$ . Observe that  $F_T(u; \beta) = F(u; \beta)$  and  $F_S(u; \beta^*) = F(u; \beta^*)$  have the same form apart from their parameters, and that  $\{F(t; \beta); \beta \in D\}$  decreases stochastically in  $\beta$ ). It follows that

$$F_S(u; \beta^*) \geq F_T(u; \beta) \quad (2.3)$$

for every  $u > 0$  and  $T \in \tau_1$ , which is equivalent to the assertion of the theorem.

Somewhat weaker results can be shown for the case that  $\sigma_0^2$  is unknown. A common practice is to estimate  $\sigma_0^2$  in a base period when the process is in control and to use this estimate in lieu of  $\sigma_0^2$  in maintaining the R and  $S^2$  charts. A similar approach may be taken with other charts. Accordingly, let  $\underline{Y}_0$  be a vector of  $m$  observations taken when the process is in control; define  $\{\underline{Y}_1, \underline{Y}_2, \dots\}$  as before; and let  $\{T(\underline{Y}_i, \underline{Y}_0); i = 1, 2, \dots\}$  be the values of a statistic to be used in a typical procedure for monitoring the process variance. For example,  $T(\underline{Y}_i, \underline{Y}_0)$  takes the values  $R_i/S_0^2$  and  $S_i^2/S_0^2$  in the modified R and  $S^2$  charts using the sample variance  $S_0^2$  from  $\underline{Y}_0$ .

Suppose the typical chart signals at level  $\alpha$  whenever  $T(\underline{Y}_i, \underline{Y}_0) \in B$ , i.e.  $P(T(\underline{Y}_i, \underline{Y}_0) \in B) = \alpha$  when the process is in control, and denote

by  $N_T$  the run length of this chart. Corresponding to (2.1) we have

$$P(N_T > k) = P(T(Y_1, Y_0) \in B^c, \dots, T(Y_k, Y_0) \in B^c). \quad (2.4)$$

The run lengths for such charts generally are not geometric even for stationary processes owing to dependencies among  $\{T(Y_i, Y_0); i = 1, 2, \dots\}$ . However, when the process is stationary but not necessarily in control, the statistics  $\{T(Y_1, Y_0), \dots, T(Y_k, Y_0)\}$  are distributed identically. The convexity of the conditional probability  $[P(T(Y_i, Y_0) \in B^c | Y_0)]^k$  as a function of  $Y_0$  and Jensen's inequality give the bound

$$P(N_T > k) \geq (1 - \beta)^k \quad (2.5)$$

with  $\beta = P(T(Y_1, Y_0) \in B)$  for all  $i = 1, 2, \dots, k$  under stationarity, which we henceforth assume. Let  $F_T(u; \beta)$  be the distribution of  $N_T$  with parameter  $\beta$  under a stationary process. Then (2.5) assures that the geometric distribution  $G(u; \beta)$  not only approximates  $F_T(u; \beta)$ , but that it is a stochastic lower bound with  $G(u; \beta) \geq F_T(u; \beta)$  for every  $u > 0$ . Moreover, equality is attained in the limit as  $m \rightarrow \infty$ , so that (2.5) is the best bound which holds for all  $m$ .

Let  $\tau_2$  be the class of all  $\alpha$ -level procedures for monitoring the process variance using statistics  $\{T(Y_i, Y_0); i = 1, 2, \dots\}$  such that  $P(T(Y_i, Y_0) \in B) > \alpha$  when  $\sigma^2 > \sigma_0^2$ . The modified  $S^2$  chart using  $S_i^2/S_0^2$  has the property that the geometric approximation to its run length is stochastically minimal among such approximations for all procedures in  $\tau_2$  when the process is stationary but not in control. Although somewhat weaker than Theorem 1, these results are summarized as follows.

THEOREM 2. Let  $\tau_2$  be the class of all  $\alpha$ -level procedures for monitoring a process variance using statistics  $\{T(\bar{Y}_i, \bar{Y}_0); i = 1, 2, \dots\}$  such that  $P(T(\bar{Y}_i, \bar{Y}_0) \in B) > \alpha$  when  $\sigma^2 > \sigma_0^2$ . If the process is a stationary Gaussian process not in control, then the approximating geometric distribution for the modified  $S^2$  chart is stochastically minimal among such approximations for all procedures in  $\tau_2$ .

Proof. For any procedure in  $\tau_2$  let  $G(u; \beta)$  be the geometric bound for the distribution of its run length, and let  $G(u; \beta^*)$  be the corresponding bound for the modified  $S^2$  chart using  $T(\bar{Y}_i, \bar{Y}_0) = S_i^2/S_0^2$ . Considered as normal-theory procedures for testing  $H: \sigma^2 = \sigma_0^2$  against  $A: \sigma^2 > \sigma_0^2$  on each sampling occasion, the class  $\tau_2$  consists of unbiased tests. The test using  $S_i^2/S_0^2$  is uniformly most powerful in this class (cf. Lehmann (1959), page 169), thus assuring that  $\beta^* \geq \beta$ . The remainder of the proof parallels that of Theorem 1.

In conclusion, Theorem 1 establishes the strict optimality of  $S^2$  charts in the class  $\tau_1$ . Theorem 2 supplies a weakly optimal property of modified  $S^2$  charts which holds for all  $m$  and is strict in the limit as  $m \rightarrow \infty$ . The interpretation is that the guaranteed stochastic lower bound is minimal among such lower bounds for all procedures in  $\tau_2$ .

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- Lehmann, E.L. (1959). Testing Statistical Hypotheses. John Wiley and Sons, Inc., New York.

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